Exercise 35

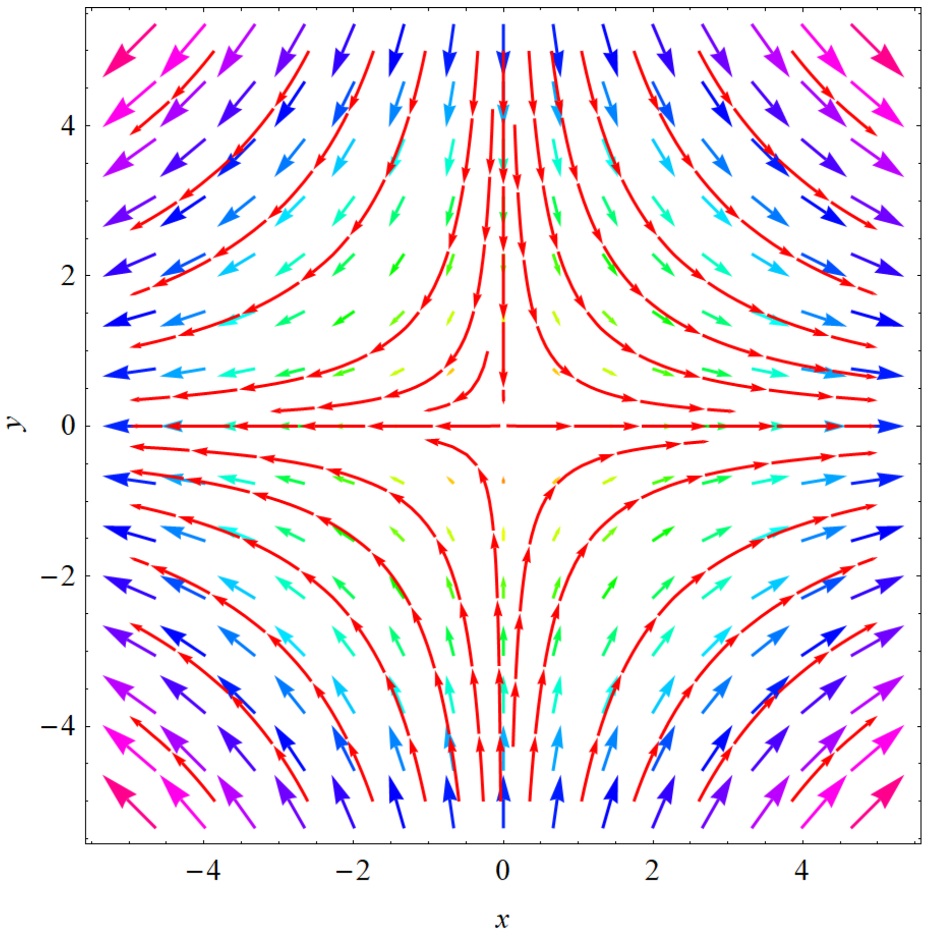
The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

(a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point $(1, 1)$.

Solution

The vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ is shown below along with several flow lines in red.



Each of the vectors in the field is tangent to a possible flow line, similar to the way a velocity vector is tangent to the corresponding position vector.

$$\mathbf{F}(x(t), y(t)) = \frac{d\mathbf{X}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

Set dx/dt equal to the x-component of $\mathbf{F}$ and set dy/dt equal to the y-component of $\mathbf{F}$.

$$\frac{dx}{dt} = x \qquad \frac{dy}{dt} = -y$$

Solving these ODEs yields

$$x = Ae^t \quad \text{and} \quad y = Be^{-t}.$$

Substitute $x/A = e^t$ into the equation for y.

$$y = B\left(\frac{A}{x}\right) = \frac{AB}{x}$$

Use a new constant C for AB.

$$y = \frac{C}{x}$$

Use the fact that the flow line has to go through $(1, 1)$ to determine C.

$$1 = \frac{C}{1} \quad \rightarrow \quad C = 1$$

Therefore,

$$y = \frac{1}{x}, \quad x > 0.$$

The restriction $x > 0$ is taken because only the part of the hyperbola that goes through $(1, 1)$ is valid.